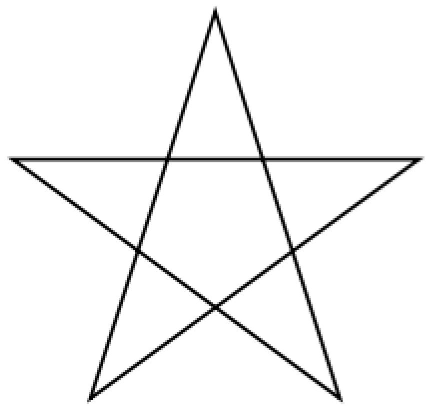
This page contains a detailed solution of Fun with Geometry: I, along with a discussion of the techniques one might use to approach the riddle. Let's first restate the problem.

Consider a regular pentagram (see picture below). There are five disjoint triangles initially. Just count the five 'caps' above the pentagon, do not count the larger triangles that have lines going through them. By adding just two lines, you can go from five triangles to ten. How?



The regular pentagram.

The statement of the problem seems straightforward, so let's just jump in. For problems like this it often helps to draw it and play and explore some possibilities. Better yet, draw it ***many*** times and try and build up some intuition. It's much faster to exhaust possibilities if the pentagram is drawn on paper and the two lines are coffee stirrers, pieces of wire, pens, or other long, thin rods which can be moved around. We'll refer to this setup later.

The first instinct is to start exhausting all of the different possibilities that come to mind. Perhaps we can stumble upon the correct solution and not have to try to find a more systematic approach to . If you're reading this solution, though, then the chances are that you've already tried 'everything' and couldn't find a solution. So perhaps a systematic approach is a good idea. The advantage of a systematic approach is that we'll keep going through all the possibilities until we find the solution. The problem is that if there are a lot of options it could take a very long time. Further, we must make sure we don't miss any options. Thus we need a good way to go through the different ways to add two lines.

STEP ONE: THINK ABOUT THE FIGURE.

Whenever you're given information in a problem, you want to use it. Let's think about the pentagram and see if that can help us figure out a good way to go through all the possibilities in an efficient way without missing anything.

The pentagram is regular; that is, the five lines are the diagonals of a regular pentagon. This ensures, among other things, that the five triangle 'caps' are all congruent, and that the pentagon within the pentagram is also regular.

Why is this important? A regular pentagram has rotational and reflective symmetry, and that will actually help us out a lot. If we have one failed attempt, we can rotate it by 72 degrees (or a multiple of 72) and get another failed attempt. This also means that there are multiple correct answers, since we can rotate a correct answer by 72 degree and get another correct answer. But does that make finding even one correct answer easier?

STEP TWO: USING ROTATIONS:

We claim that we can use the symmetry of the pentagram to narrow down our search. To see how, we can first fix the position of one of our two lines (wires, coffee stirrers, pens, etc.). For instance, we could fix a vertical line that passes through the top vertex of the pentagram. Then we could try to exhaust all possible places to put the second line and see if any of these places gives us ten triangles. If none of these positions give a correct answer, and if we've truly exhausted all possible choices for the second line, then we will have to remove the second line and reposition the first line. But now where do we place the first line? The point is that there are a few positions for the first line that we do not need to check. Namely, we do not need to try placing the first line in its original place rotated by a multiple of 72 degrees! If we do, then any position for the second line has to give an incorrect answer, since it gives an incorrect answer rotated by a multiple of 72 degrees.

STEP THREE: USING REFLECTIONS:

So far we've just used rotation, but we can also use reflection. If we think again about the vertical line passing through the top vertex of the pentagram, if we placed our second line unsuccessfully ad then reflected it across the first line (you can do that even if the second line lies on both sides of the first line), we would get another unsuccessful attempt.

So if we think of our problem as searching a space of possibilities, then what we've done is use symmetry of the pentagram to make our search more efficient. (If we think of our space of possibilities as an actual space like the one we live in, where the points of the space represent different ways of placing the lines, then what we've done is fold the space. We've done this by gluing together points that represent "equally correct" or "equally incorrect" line placements based on the symmetry of the pentagram. If this doesn't make any sense, that's okay. All we need to know is that we've narrowed down all of the different things we can try.)

There are more ways to make our search more systematic, and it is unclear which way gives us an answer more quickly (or at all). There may, in fact, not even be a best approach. Let's see what works.

STEP FOUR: SYSTEMATIC SEARCHING:

Idea #1: Fix the point where the lines intersect and then rotate the lines around it.

Intuitively, a correct answer should have the lines intersect the pentagram in many places. It is reasonable to think that the more intersections that a bunch of lines make, the more triangles there are on average. So in particular we would expect for a correct solution that the two lines that we place down intersect themselves. So what we can do is pick a point on, outside of, or inside of the pentagram where we think the lines are going to intersect. Then we're going to place one line at a time on that dot and rotate it and see how many new intersections we can make.

Now, let's fix our point of intersection but only focus on placing one line. Here's a zone of all possible places we should check.

Right? Because any other point can be reflected across a few of the pentagram's lines of symmetry in succession (possibly just one line of symmetry) to get a point in the orange zone. Therefore, one solution attempt can be reflected in the same way to give a solution attempt that is as correct as the original attempt. So it suffices to restrict our point of intersection to the orange zone.

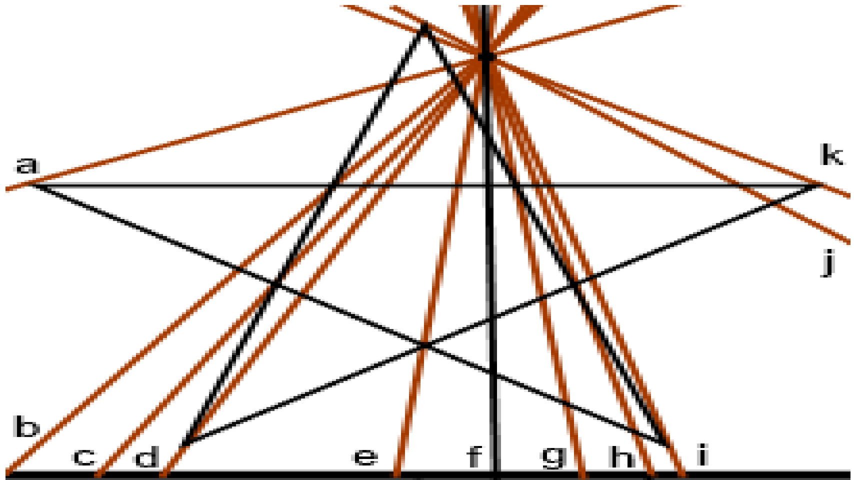
This may not narrow things down quite as much as we'd hope. It turns out that placing the point of intersection right at the top tip of the pentagram gives a different possible set of arrangements than placing the point very very close to the tip. But it does narrow things down.

Let's pick our point to be outside the pentagram. Let's also place a vertical line passing through said point.

What happens (in terms of the number of triangles or number of intersections) when we rotate this line counterclockwise? This is easy to do with wires or coffee stirrers.

We notice that there are specific points at which the number of intersections/triangles changes. This happens precisely when the line “passes over” a vertex of the pentagram. The number of intersections/triangles can be different before, after, or while the line is “on” the vertex of the pentagram. But the number of intersections or triangles does not change anywhere else.

The data of number of intersections/triangles is summarized in the picture below.



If the bottom of the line drawn is...	# of inter-sections	# of triangles
left of a or right of k	2	5
at a	3	6
between a and b	4	6
at b	3	7
between b and c	4	6
at c	3	6
between c and d	4	5
at d	4	6
between d and e	4	5
at e	3	5
between e and g (no typo!)	4	5
at g	3	6
between g and h	4	6
at h	3	7
between h and i	4	6
at i	3	6
between i and j	2	5
at j	2	6
between j and k	4	6
at k	3	6

Observe that the line passing through the point of intersection and j also intersects the top vertex of the pentagram.

STEP FIVE: INTERPRET AND CONTINUE SEARCHING:

As the picture shows, our 180 degrees of freedom can be broken into "segments" based on the shape that the line makes with the pentagram. These "segments" are represented by the table by phrases such as "between h and i" and "at b." Two lines whose bottom points are in the same "segment" create the same number of intersections and triangles. For instance, two lines whose bottom points both lie between c and d in the picture both make four intersections with the pentagram and make for a total of five triangles. It is important to recognize that some segments are just points. (That is, some possible configurations are achieved only when the line is placed at one particular slope.) On the other hand, some segments can be fairly wide.

Now what we do is we pick one slope for the first line. Then we place the second line, rotate just the second line, and see if we get ten triangles. If that fails, we take off the second line and rotate the first line until the slope falls within a different segment. Then we try the second line again. If that fails, set the first line in another segment, and so on.

As we mentioned before, the difficulty with this approach lies in exhausting all possible points of intersection. For instance, it may be tempting to think that all points of intersection from the orange zone outside the pentagram give equal possibilities. But this is not true.

We spoke earlier about placing the first line and seeing how well other lines go with it. Perhaps if the first line is placed strategically, then we'll be able to place the second line and make ten triangles.

Idea #2: Place your first line "well enough."

The real battle is determining what placing your first line well enough means. With our first line, do we want to increase the total number of triangles by as many as possible? Do we want to intersect the pentagram in as many points as possible? The second approach actually looks more promising. As we mentioned earlier, if we want our seven lines (five from the pentagram, two that we place) to form ten triangles, then we would want lots of intersections.

What's the greatest number of points that one line can intersect the pentagram? We can find a few examples that intersect the pentagram in four points. We don't need to find all examples, but hopefully we can find most of the examples of one line making four intersections (there aren't too many). Remember that we can use symmetry to cut down the number of unique examples by a lot.

What about five intersections? Any line not parallel to one of the five lines of the pentagram will intersect all five of those lines- IF we extend them. But we're not allowed to do that. So perhaps it's not possible to intersect all five lines. For the purposes of this problem, we don't need to prove that.

Let's just take our examples with four intersections and test them one by one, adding the second line and trying to make as many intersections as possible. In what is perhaps the ideal situation, our two lines should intersect each other in one point, and they should intersect the pentagram in four points each. If we play around a bit, keeping that goal in mind, we get to this solution:

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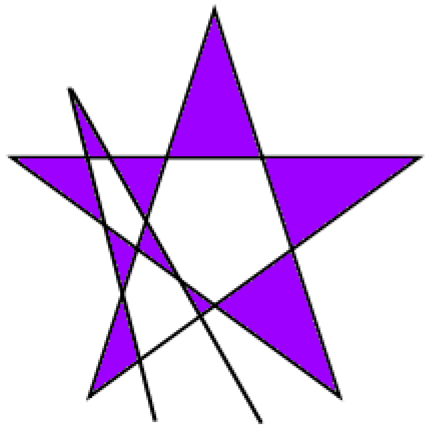

GENERALIZING THE PROBLEM:

How do we generalize this problem? We could increase the number of lines, trying to maximize the number of triangles with three lines, then four, and so on. We can try to use our winning strategy: try to have as many intersections as possible. As the number of lines increases, though, this will become unreasonably complex to do by hand. We could try to guess and prove a formula for the maximum number of triangles given by n lines. We could also try to draw lines over other shapes, including (but not limited to) stars with six or more points.

CONCLUDING THOUGHTS:

The main purpose of this problem was to learn how to search a space of possibilities effectively. Spaces of all ways to place objects are called configuration spaces and are fun to think about, and many of these spaces are the subject of scholarly mathematical research. It may be fun to think of various types of configuration spaces and what they look like- for instance, the space of all possible ways to place two points labeled 1 and 2 on a line segment so that 1 is to the left of 2. That space turns out to be a triangle- can you see it?

Happy hunting!